N-Soliton Train Interactions and Perturbed Complex Toda Chain in Nonlinear Optics.

Adiabatic and non-adiabtic aspects.

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Abstract

Our previous results on the N-soliton interaction in the adiabatic approximation have been extended. It is shown that the complex Toda chain (CTC) model is an universal one in the sense that it describes the N-soliton train interactions for all NLEE from the NLS hierarchy. We derive the perturbed CTC system and show that the small perturbations affect only the center of mass motion and the global phase of the N-soliton train. A special reduction of CTC describes the interaction of the sine-Gordon solitons and anti-solitons. The peculiarities of the interactions in the non-adiabatic cases are outlined.

Keywords: optical solitons; nonlinear guided waves; optical communication systems.

1 Introduction

A number of nonlinear optical phenomena in Kerr-like media [1] are described by the perturbed nonlinear Schrödinger equation (NLSE) which in dimensionless variables reads

$$i\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + |u|^2 u(x, t) = iR[u], \tag{1}$$

and by its multicomponent generalizations. Below we shall discuss several perturbations $R[u]$ due to possible linear losses, bandwidth limited and nonlinear amplifications.

For $R[u] = 0$ the NLSE (1) can be solved by means of the inverse scattering method [2, 3] applied to the Zakharov-Shabat system $L[u]$. The analytical methods developed allow one: i) to prove that any solution of the NLSE in the limit $t \to \infty$ tends to a purely solitonic solution; ii) describe the soliton interaction in the generic case when all solitons have pair-wise different velocities.

For practical applications one needs to describe the behavior of the so-called N-soliton trains which are solutions of (1) satisfying the initial condition

$$u(x,0) = \sum_{k=1}^{N} u_k^{1s}(x, t = 0). \tag{2}$$

Here $u_k^{1s}(x,t)$ is the one-soliton solution of (1):

$$\begin{aligned}
u_k^{1s}(x,t) &= 2\nu_k e^{i\phi_k} \operatorname{sech}(z_k(x,t)), & z_k(x,t) &= 2\nu_k(x - \xi_k(t)), \\
\phi_k(x,t) &= 2\mu_k(x - \xi_k(t)) + \delta_k(t), & \xi_k(t) &= 2\mu_k t + \xi_{k,0}, \\
\delta_k(t) &= 2(\mu_k^2 + \nu_k^2)t + \delta_{k,0},
\end{aligned} \tag{3}$$

where ν_k, μ_k, ξ_k and δ_k are the amplitude, velocity, position and phase of the k-th soliton-like pulse.

Let us first remark that for $N \geq 2$ the parameters μ_k and ν_k *are not* directly related to the discrete spectrum of $L[u]$. In fact the spectral data of $L[u]$ with u provided by (2) contains not only $2N$ discrete eigenvalues $\lambda_k^{\pm} = \kappa_k \pm i\eta_k$, $\eta_k > 0$, $k = 1, \ldots, N$, but also nonvanishing 'radiation' related to the continuous spectrum of $L[u]$. However if we take well separated pulses $|\xi_{k+1,0} - \xi_{k,0}| \simeq r_0 \gg 1$ then the energy of the 'radiation' is of the order of 1% of the total energy and may well be neglected. As a result the corresponding N-soliton train may be approximated by an N-soliton solution whose interactions in the generic case (pair-wise different velocities) are well known [2, 3]. Even if we approximate $u(x,t)$ by an exact N-soliton solution it is not an easy matter to evaluate the discrete eigenvalues $\lambda_k^{\pm}$ and the corresponding 'normalization' constants $C_k^{\pm}$ of the Jost solutions; this is easy only in the limit $r_0 \to \infty$, when $\lambda_k^{\pm} = \mu_k \pm i\nu_k$.

Other difficulties come from the fact that in many of the applications we need to analyze trains in which: a) the solitons move with nearly the same velocities; b) various perturbations should be taken into account. In such cases the exact approach based on the inverse scattering method [2, 3] can not be directly applied and one should look for other methods [4, 5].

Our aim is to extend our previous results on the N-soliton interaction in the adiabatic approximation [4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. Firstly we show that the CTC model is an universal one in the sense that it describes the N-soliton interactions for all NLEE from the NLS hierarchy. We derive the perturbed CTC (PCTC) system and show that small perturbations affect only the center of mass motion and the global phase of the N-soliton train. Special attention is paid to the (anti-) soliton interaction of the sine-Gordon equation. We show that in the adiabatic approximation their interaction is described by the Toda chain with indefinite metric, which is a special reduction of the CTC. We also outline the peculiarities of the interactions in the non-adiabatic cases.

This paper is an extended version of [14].

2 The N-soliton interactions and the Complex Toda Chain

In [8] the quasi-particle approach of Karpman and Solov'ev has been generalized to any $N > 2$ soliton train in the adiabatic approximation. This means that the solitons initially must have nearly equal amplitudes and velocities and must be well separated, i.e.:

$$|\nu_{k,0} - \nu_{j,0}| \ll \nu_0, \qquad |\mu_{k,0} - \mu_{j,0}| \ll \mu_0,$$
$$\nu_0(\xi_{k+1,0} - \xi_{k,0}) \simeq r_0 \gg 1; \qquad |\nu_{k,0} - \nu_0|(\xi_{k+1,0} - \xi_{k,0}) \ll 1, \qquad (4)$$

where the additional zeroes in the subscripts in (4) refer to the value at $t = 0$ and ν_0 and μ_0 are the average amplitude and velocity of the N-soliton train. The result is a dynamical system of equations for the $4N$ soliton parameters called the generalized Karpman-Solov'ev system (GKS).

The GKS is adapted to treat also the perturbed NLS equation. An exhaustive list of perturbations, relevant for nonlinear optics, which include linear and nonlinear dispersive and dissipative terms, effects of sliding filters, amplitude and phase modulations, etc. is studied [10, 11, 13]. We prove that the linear perturbations affect each of the solitons separately, while the nonlinear ones lead to additional interactive terms between neighboring solitons.

Another important step which allowed us to analyze the N-soliton interactions analytically consists in the fact, that under some additional approximations the GKS reduces to the complex Toda chain (CTC) with N nodes [9, 10]:

$$\frac{d^2 q_k}{dt^2} = 16\nu_0^2 \left(e^{q_{k+1} - q_k} - e^{q_k - q_{k-1}} \right), \qquad (5)$$

where $k = 1, \ldots, N$ and we assume that $e^{-q_0} \equiv e^{q_{N+1}} \equiv 0$. The complex dynamical variables $q_k(t)$ are expressed in terms of the parameters of the k-th soliton by:

$$q_k(t) = 2i(\mu_0 + i\nu_0)\xi_k(t) - i(\delta_k(t) + \delta(t)) + kQ_0,$$
$$Q_0 = \ln 4\nu_0^2 + i\pi, \qquad \delta(t) = \frac{1}{N} \sum_{s=1}^{N} \delta_k(t). \qquad (6)$$

The same result has been derived also by using the variational approach [7]; this approach however should be applied with care, see [15].

The CTC with N nodes, which may be viewed as a natural generalization of the corresponding real Toda chain (RTC), provides a very convenient tool to study the soliton interactions. Numeric simulations show that CTC provides an adequate description for the soliton interactions for a wide class of initial conditions (2) [10, 11, 13].

It is also possible to describe the soliton interactions of other soliton-type nonlinear equations with different dispersion laws. For any such equation one

can derive the corresponding GKS and CTC model which could be useful to study the interactions of their N-soliton trains, see [16].

The adiabatic approximation imposes certain restrictions not only on the soliton parameters (see Eq. (4)), but also on the spectral data of the Zakharov-Shabat system L. Firstly the discrete eigenvalues in the upper λ-half plane of L must be located in a small neighborhood around λ_0:

$$|\lambda_k^+ - \lambda_0|^2 \simeq \varepsilon, \qquad \lambda_0 = \sum_{k=1}^{N} \lambda_k^+ / N, \qquad (7)$$

where the small number ε determines the overlap between the neighboring solitons. With the same precision the eigenvalues λ_k^+ can be approximated by $2\zeta_k$ where ζ_k are the eigenvalues of the Lax matrix for the CTC. Secondly we have a condition on the initial values of the constants C_k which determine the initial positions and phases of the pulses. Skipping the details we get:

$$\ln \left| C_{k+1}^+(0)/C_k^+(0) \right| = 2\nu_0(\xi_{k+1,0} - \xi_{k,0}) + \mathcal{O}(1) \simeq -2\ln\varepsilon \gg 1. \qquad (8)$$

The conditions on the discrete eigenvalues λ_k^+ allow us also to explain the universality of the CTC as a model describing the N-soliton interactions. Namely, we claim that the CTC model describes in the adiabatic approximation the N-soliton interactions of all NLEE in the NLS hierarchy. Indeed, let us consider a higher NLS equation with dispersion law $F(\lambda)$, regular in the vicinity of λ_0. Then the time-dependence of C_k^+ is given by

$$C_k^+(t) = \exp(2iF(\lambda_k^+)t)C_k^+(0). \qquad (9)$$

As a consequence the one-soliton solution will be given by (3) with $z_k(x,t)$ and $\delta_k(t)$ replaced by

$$z_k(x,t) = 2\nu_k \left(x - \frac{f_{1,k}}{\nu_k}t - \xi_{0,k} \right), \qquad (10)$$

$$\delta_k(t) = \delta_{k,0} + \frac{2(\mu_k f_{1,k} - \nu_k f_{0,k})}{\nu_k}t, \qquad (11)$$

where $F(\lambda_k^+) = f_{0,k} + if_{1,k}$. However, due to (7) in fact it is enough to take into account only the first three terms in the Taylor expansion:

$$F(\lambda_k^+) = F(\lambda_0) + (\lambda_k^+ - \lambda_0)\dot{F}_0 + \frac{1}{2}(\lambda_k^+ - \lambda_0)^2\ddot{F}_0 + \mathcal{O}(\varepsilon^{3/2}), \qquad (12)$$

where $\dot{F}_0 = (dF/d\lambda)|_{\lambda=\lambda_0}$, $\ddot{F}_0 = (d^2F/d\lambda^2)|_{\lambda=\lambda_0}$. Comparing (9) and (10) we see that in the adiabatic approximation only the first three terms in (12) are important for the soliton parametrization. This explains why the soliton interactions for all the equations from the hierarchy is described by the same

4

universal model: CTC. As an example of higher NLS equations which also finds important applications in nonlinear optics is the one with dispersion law $F(\lambda) = 2\lambda^2 + \eta\lambda^3$ introduced in [17]; the GKS for this equation is derived in [16]. The corresponding CTC model is obtained from (5) by replacing the coefficient $16\nu_0^2$ with a factor depending on $F(\lambda_0)$ which can be taken care of by redefining q_k. Another possibility is to choose $F(\lambda) = 1/(2\lambda)$ which after additional reduction leads to the sine-Gordon equation, see Section 4 below.

These results have been further developed by using the fact, that the CTC is a completely integrable dynamical system with $2N$ degrees of freedom. The most important consequence of this fact lies in the possibility to predict the asymptotic behavior of the solitons from the set of their initial parameters [9, 12]. Indeed, knowing the initial soliton parameters we can construct the eigenvalues of the Lax matrix for the CTC system, which in turn determine the asymptotic velocities of the solitons.

A more detailed study of the solutions of the CTC allowed us to see that it allows much richer class of asymptotic regimes than the RTC [12, 13]. We are also able to describe the class of initial soliton parameters, that lead to each one of these regimes: i) asymptotically free propagation of the solitons (the only regime allowed by RTC); ii) N-soliton bound states with the possibility of a quasi-equidistant propagation; iii) mixed asymptotic regimes when part of the solitons form bound state(s) and the rest separate from them; iv) regimes corresponding to the degenerate and singular solutions of the CTC.

In [9]–[11], [13] a thorough comparison between the CTC predictions with the numerical solutions of the NLS equation has been performed and an excellent match has been established for a number of choices of the initial soliton parameters in each of the regimes listed above. Special attention has been paid to regime ii) and more specifically to the possibility for a quasi-equidistant (QED) propagation of all N solitons. A method for the description of the corresponding initial soliton parameters responsible for this regime has been proposed.

3 Perturbed NLS and the perturbed CTC

In [10] we showed also that the evolution of the N-soliton train (2) of the perturbed NLS equation (1) is described by the following dynamical system for the 'slow' evolution of the soliton parameters:

$$\frac{d\nu_k}{dt} = 16\nu_0^2(S_k - S_{k+1}) + N_k, \tag{13}$$

$$\frac{d\mu_k}{dt} = -16\nu_0^2(C_k - C_{k+1}) + M_k, \tag{14}$$

$$\frac{d\xi_k}{dt} = 2\mu_k + \Xi_k^{(0)} + \Xi_k, \tag{15}$$

$$\frac{d\delta_k}{dt} = 2(\mu_k^2 + \nu_k^2) + X_k^{(0)} + X_k, \tag{16}$$

5

where

$$\Xi_k^{(0)} = -4(S_k + S_{k+1}),\tag{17}$$

$$X_k^{(0)} = 2\mu_k\Xi_k^{(0)} + 24\nu_k(C_k + C_{k+1}),\tag{18}$$

$$C_k(t) - iS_k(t) = -\frac{1}{4\nu_0}e^{q_k(t)-q_{k-1}(t)}.\tag{19}$$

The terms $N_k, \ldots, X_k$ are determined by $R[u]$ below. As it was shown in [10] they contain two types of terms: a) 'self-interaction' terms depending only on the parameters of the k-th soliton and b) 'nearest-neighbour' interaction terms containing linear combinations of S_k, C_k, S_{k+1} and C_{k+1}.

In [10] we also derived the explicit expressions for $M_k, \ldots, X_k$ in terms of the soliton parameters for several classes of physically important perturbations. Here we take into account linear and cubic in u perturbations including the linear and nonlinear gain, third order dispersion (TOD), intrapulse Raman scattering (IRS) etc, i.e.:

$$R[u] = \sum_{k=0}^{3} c_k\frac{\partial^k u}{\partial x^k} + d_0|u|^2 u + \frac{d_1}{4}u(|u|^2)_x + \frac{d_2}{4}(|u|^2 u_x - u_x^* u^2),\tag{20}$$

where c_s and d_s are generically complex parameters:

$$c_s = c_{s0} + ic_{s1}, \qquad d_s = d_{s0} + id_{s1}.$$

Some of these coefficients, namely c_{01}, c_{21} and d_{01} can be put to zero without restrictions; this can be done by conveniently renormalizing u, t and x.

The next argument which we will use is that the coefficients in (20) must be small. We start by assuming that they are, like the terms S_k and C_k, of the order of ε; at the same time the deviations $\tilde{\nu}_k = \nu_k - \nu_0$, $\tilde{\mu}_k = \mu_k - \mu_0$ are of the order of $\sqrt{\varepsilon}$. Therefore in the right hand sides of the equations (13)-(14) we have only terms of the order of ε, while in the r.h.sides of (15)-(16) we have also terms of the order of 1 and $\sqrt{\varepsilon}$. That is why we will simplify the perturbative terms in the r.h.sides of (13)-(16) by taking only the first few terms in their Taylor expansions, i.e.

$$Z_k(\nu_k, \mu_k) = Z_{00} + \tilde{\nu}_k Z_{10} + \tilde{\mu}_k Z_{01}, \qquad Z_{00} = Z(\nu_0, \mu_0),$$

$$Z_{10} = \frac{\partial Z}{\partial\nu_k}\bigg|_{\substack{\nu_k=\nu_0\\\mu_k=\mu_0}}, \qquad Z_{01} = \frac{\partial Z}{\partial\mu_k}\bigg|_{\substack{\nu_k=\nu_0\\\mu_k=\mu_0}},\tag{21}$$

where Z stands for each of the functions N_k, M_k, Ξ_k and X_k. The explicit expressions for the coefficients in (21) for each of the four functions are given in the appendix.

3.1 Perturbations of order ε.

Note that due to our assumption about the perturbation constants all coefficients in Z_{00} are of the order of ε; so in fact we have to take into account only N_{00} and M_{00}. As a result we derive the following perturbed version of the CTC model:

$$\frac{d^2 q_k}{dt^2} = U_{00} + 16\nu_0^2 \left(e^{q_{k+1}-q_k} - e^{q_k - q_{k-1}} \right), \qquad (22)$$

where $U_{00} = -4\mu_0 N_{00} - 4\nu_0(M_{00} + 2iN_{00})$. In deriving (22) we took into account also the fact that now $\lambda_0 = \sum_{k=1}^{N} \lambda_k / N$ and consequently ν_0 become time-dependent:

$$\frac{d\lambda_0}{dt} = \frac{1}{N} \sum_{k=1}^{N} (M_k + iN_k) \simeq M_{00} + iN_{00}, \qquad \frac{d\nu_0}{dt} \simeq N_{00}.$$

Eq. (22) can be solved exactly with the result:

$$q_k(t) = \frac{1}{2} U_{00} t^2 + V_{00} t + q_k^{(0)}(t), \qquad (23)$$

where $q_k^{(0)}(t)$ is a solution of the unperturbed CTC and V_{00} is an arbitrary constant.

If in addition we assume that $V_{00} = 0$ and $\mu_0 = 0$ then from the formulae in the appendix we find that:

$$U_{00} = \frac{16\nu_0^2}{3} \left\{ \nu_0 \left[c_{11} + \frac{4}{5}\nu_0^2(d_{11} - 7c_{31}) \right] - i \left[3c_{00} + 4\nu_0^2(2d_{00} - c_{20}) \right] \right\}. \qquad (24)$$

We remind that $q_k^{(0)}$ is related to the k-th soliton parameters by (6). Then from (23) we see that $\mathrm{Re}\, U_{00}$ and $\mathrm{Re}\, V_{00}$ influence the center of mass motion of the train while $\mathrm{Im}\, U_{00}$ and $\mathrm{Im}\, V_{00}$ drive the global phase $\delta(t)$. In particular for the special case when $\mathrm{Re}\, U_{00} = 0$ the effect of such perturbation will be to make the phase of all solitons oscillate simultaneously with a rate proportional to t^2. However the evolution of the phase and coordinate differences $\delta_{k+1} - \delta_k$ and $\xi_{k+1} - \xi_k$ will not be influenced.

3.2 Perturbations of order $\sqrt{\varepsilon}$.

More complicated and substantially different is the situation when the perturbative constants become of order $\sqrt{\varepsilon}$. Then the terms N_k, M_k in (13), (14) can be approximated by linear combinations of $\tilde{\nu}_k$, $\tilde{\mu}_k$; as a result Equation (13)

acquires the form:

$$\frac{d\nu_k}{dt} = N_{00} + N_{10}\tilde{\nu}_k + N_{01}\tilde{\mu}_k + 16\nu_0^2(S_k - S_{k+1}), \tag{25}$$

$$\frac{d\mu_k}{dt} = M_{00} + M_{10}\tilde{\nu}_k + M_{01}\tilde{\mu}_k - 16\nu_0^2(C_k - C_{k+1}), \tag{26}$$

$$\frac{d\xi_k}{dt} = 2\mu_k + \Xi_{00} + \Xi_{10}\tilde{\nu}_k + \Xi_{01}\tilde{\mu}_k + \Xi_k^{(0)}, \tag{27}$$

$$\frac{d\delta_k}{dt} = 2(\mu_k^2 + \nu_k^2) + X_{00} + X_{10}\tilde{\nu}_k + X_{01}\tilde{\mu}_k + X_k^{(0)}, \tag{28}$$

Then N_{00} and M_{00} (of order $\sqrt{\varepsilon}$) will be the leading order terms in (25), (26) while terms like $c_0\tilde{\nu}_k$, $c_2\tilde{\mu}_k$ are of the same order ε as the interaction terms (ones with S_k and C_k). The solutions of such PCTC will be qualitatively different from the ones of CTC and require separate studies.

4   Sine-Gordon solitons and the CTC

Here we shortly discuss the (anti-) soliton interactions of the sine-Gordon equation:

$$v_{xt} + \sin v(x,t) = \varepsilon R[v]. \tag{29}$$

The problem has been attacked long time ago by Spatschek [18] and Karpman and Solov'ev [4] where the interaction of two (anti-) solitons has been studied. The one-soliton solution is given by

$$v^{1s} = 2\sigma \arcsin \tanh z \pm \pi, \qquad z = 2\nu(x - \xi(t)),$$

where $\xi(t) = t/(4\nu^2) + \xi_0$. For $N > 2$ and $R[v] = 0$ in the adiabatic approximation only the nearest neighbor interactions are relevant. Then the results of [18, 4] generalize to:

$$\frac{d\nu_k}{dt} = 4\left(e^{Q_{k+1}-Q_k} - e^{Q_k-Q_{k-1}}\right), \tag{30}$$

$$\frac{d\xi_k}{dt} = \frac{1}{4\nu_k^2} + \frac{1}{\nu_0^2}\left(e^{Q_k-Q_{k-1}} + e^{Q_{k+1}-Q_k}\right), \tag{31}$$

$$Q_k(t) = -2\nu_0\xi_k(t) + \frac{i\pi}{2}(1 - \sigma_k), \tag{32}$$

where $\sigma_k = 1$ (or -1) if at position k we have soliton (or anti-soliton). Note that the sine-Gordon (anti-) solitons do not have internal degrees of freedom and are characterized only by their amplitudes ν_k and positions ξ_k; here we choose $\xi_1 < \xi_2 < \cdots < \xi_N$ and ν_0 is the average amplitude.

We remark also that the sine-Gordon equation is related to the Zakharov-Shabat system $L[u]$ if we assume $u = v_x/2$ and request that v is real. This last

condition has important consequences: i) besides the (anti-) soliton solutions related to the purely imaginary eigenvalues $\lambda_k^\pm = \pm i\nu_k$ of L, the sine-Gordon equation has also breather solutions; ii) one can not have two (anti-) solitons moving with the same speed, i.e. $\nu_k \neq \nu_j$ for $k \neq j$.

The adiabatic approximations (7) mean that $|\nu_k - \nu_{k+1}| \simeq \mathcal{O}(\sqrt{\varepsilon})$ and in addition we have (8). Thus we see that in the right hand side of (31) only the first term is the relevant one; the other two are of the order of $\mathcal{O}(\varepsilon)$ and can be neglected. If we now differentiate (31) with respect to t, use (30) and keep only terms of order ε we get:

$$\frac{d^2 Q_k}{dt^2} = \frac{4}{\nu_0^2} \left(e^{Q_{k+1} - Q_k} - e^{Q_k - Q_{k-1}} \right), \tag{33}$$

where we replaced ν_k in the denominator by ν_0.

If all σ_k are equal we obtain the real Toda chain. It is also known from the spectral properties of the Zakharov-Shabat system, that in the sine-Gordon case we can not have two (anti-) solitons moving with the same velocities. This means that if we have a sequence of solitons (or anti-solitons) only then their interaction is purely repulsive and their asymptotic regime can contain only asymptotically free 'particles'. This facts are compatible with the analytical results on the sine-Gordon solitons, see [2, 3].

The model (33) with generic σ_k has been studied by Kodama and Ye [19] and is known as the Toda chain with indefinite metric. In both cases we can view (33) as special reduction of the CTC. Thus we see that the second involution on the Zakharov-Shabat needed for the sine-Gordon equation, carries over as a reduction on the CTC. We should also note that the equations (32), (33) have solutions with singularities which are periodic in time, see [19, 12]. The comparizon between the N (anti-) soliton train dynamics of the sine-Gordon equation and the indefinite metric Toda chain is yet to be done. This and the studies of the PCTC for the perturbed sine-Gordon equation will be published elsewhere.

5 Non-adiabatic Interactions

If one or more of the 'adiabatic conditions' (4) are violated then the picture becomes much more complicated. It is possible that due to strong perturbation some of the soliton pulses come very close to each other and strongly overlap. Usualy this is combined with strong deformations of the pulses and substantial emission of 'radiation' which is not accounted for in our model.

To our knowledge there are no effective models which would provide analytic description of the soliton interactions in such situations. As main tool giving a physical insight of the soliton dynamics is the comparison between the numerical solutions to the NLS equation (1) and the numerical solution of the corresponding Zakharov-Shabat spectral problem [20].

The method consists in the following: first we solve numerically the corresponding (perturbed) NLS equation using the standard fast-Fourier transform (or beam-propagation) method. Then we use the results for the pulse shape evaluated at a certain distance as an initial potential for the Zakharov-Shabat eigenvalue problem and determine numerically its scattering data [21]. As a result we can determine the time evolution of the scattering data (including the data, characterizing the continuous spectrum).

The advantage of this method is the possibility to follow up the variations of the amplitudes and the velocities of an arbitrary number of solitons. Here it is possible in a natural way to estimate the energy of the 'radiation', related to the continuous spectrum of L. The disadvantage is in the necessity to know approximately the locations of the eigenvalues of L at $t = 0$.

Below we use the basic fact that the unperturbed NLS equation is integrable. As a consequence the evolution of $u(x, t)$ preserves the spectrum of corresponding Zakharov-Shabat system L, which may be determined from the initial condition $u(x, t = 0)$. In particular the discrete eigenvalues of L will be time-independent since they are integrals of motion of the NLSE.

If we next consider perturbed NLS equation then generically the perturbation will violate the integrability. However we assume that the perturbation is 'small' in the sence that it does not destroy completely the integrability but rather slightly modifies the spectrum of L. In particular the eigenvalues $\lambda_k^{\pm}$ of L start to move; here and below the upperscript $+$ $(-)$ means that the corresponding eigenvalue is such that $\mathrm{Im}\,\lambda_k^+ > 0$ ($\mathrm{Im}\,\lambda_k^- < 0$). We remind that the involution $\lambda_k^+ = (\lambda_k^-)^*$ holds, so it is enough to know only the discrete eigenvalues λ_k^+.

To our knowledge there are no explicit criteria which would allow one to check whether given perturbation is 'small' or not. Some inexplicit criteria have been formulated in [22]; in particular they require that the eigenvalues λ_k^+ remain in the upper half-plane (i.e. $\mathrm{Im}\,\lambda_k^+ > 0$ for all t), that they do not come close to the real axis and that they do not coalesce. In terms of the soliton parameters the second of these condition means that the amplitude of the pulses should not become very small.

In [20] an investigation of the influence of the intrapulse Raman scattering and the third order dispersion on the discrete eigenvalues of the Zakharov-Shabat system have been performed by numerical means. Two qualitatively different initial conditions approximating two-soliton bound states have been studied.

The first one $u_1(x, t = 0) = 2\mathrm{sech}\,(x)$ correponds to strongly overlapped soliton pulses; so in this case the adiabatic approximation is not valid. The spectrum of L_1 (i.e. of L with potential given by $u_1(x, t = 0)$) consists of two eigenvalues in the upper half-plane [23] with $\lambda_k^+ = i(k - 1/2)$, $k = 1, 2$ (and two more in the lower half-plane). Note also that the distance between these eigenvalues is not small.

For the second one $u_2(x, t = 0) = \mathrm{sech}\,(x - \delta) + \mathrm{sech}\,(x + \delta)$ with $\delta \simeq$

$3 \div 4$ the pulses are well separated and the adiabatic approximation holds. The spectrum of L_2 can not be calculated presicely; besides the two pairs of eigenvalues it contains also some small 'radiation'. The eigenvalues λ_k^+ can be well approximated by the eigenvalues of the Lax matrix for the corresponding CTC. In our case this give:

$$\lambda_1^+ \simeq \frac{i}{2}\left(1 + e^{-\delta}\right) \qquad \lambda_2^+ \simeq \frac{i}{2}\left(1 - e^{-\delta}\right). \tag{34}$$

Note that already for $\delta \simeq 3 \div 4$ the quantity $e^{-\delta}$ may be considered as small (of the order of $\sqrt{\varepsilon}$; i.e., these eigenvalues satisfy the adiabaticity condition. Obviously, if take δ to be smaller then the overlap of the solitons grows and the adiabaticity is violated. As a result (34) does not give correct values for the eigenvalues. In the limit of infinitely separated soliton pulses, i.e. $\delta \to \infty$ the eigenvalues (34) coalesce. This is related to the fact that the solution to the CTC with these initial conditions is singular, see [12].

The effect of these two perturbation on the eigenvalues of L are similar for both types of initial conditions. In what follows we describe it for the non-adiabatic case with $u_2(x, t = 0)$.

First we analyse the effect of the third order dispersion (TOD) which is a Hamiltonian perturbation. The time evolution of the eigenvalues is shown on Fig. 1 for different strengths c_{30} of TOD. For $c_{30} \leq 0.01$ it turnes out that the imaginary parts are almost constant while the real parts are zero.

It is known also that there exist a critical value $c_{30,\text{cr}} = 0.022$ [24] where the two-soliton bound state breaks down. For $c_{30} = 0.02$, which is just below the critical value two strongly fluctuating real parts show up. Minor fluctuations of the imaginary parts can also be identified. For $c_{30} = c_{30,\text{cr}}$ the very splitting of the degenerate real parts $\kappa_1 = \kappa_2 = 0$ appears. After some transition time both real parts attain constant but different values. The imaginary parts remain almost unchanged. This stage of deformation of the eigenvalues caused by TOD corresponds to the break up of the two-soliton bound state into two single, progressively separating solitons with amplitudes determined by the initial imaginary parts of the eigenvalues. This behavior of the eigenvalues is consistent with the previous results.

If c_{30} grows even larger (e.g., $c_{30} \geq 0.03$), the smaller imaginary part changes significantly. This second stage of deformation of the eigenvalues can be described by the ultimate differences between both real and imaginary parts, respectively, which increase with c_{30}. The change in the imaginary part leads to the creation of 'radiation'.

A similar investigation was performed in order to analyze the effect of dissipative perturbation such as intrapulse Raman scattering on the eigenvalues. The results are shown in Fig. 2. A remarkable fact to be mentioned is that unlike for TOD a very weak perturbation ($d_{11} = -0.0004$) lifts the degeneracy of the real part. The two-soliton bound state breaks up and two slowly separating solitons with different but constant amplitudes emerge. The second stage in

11

the deformation (changes in the imaginary parts) start from $d_{11} = -0.02$ and differs from the TOD case in that both imaginary parts change. In contrast to TOD the effect shows up for considerably weaker perturbations.

The big change of the larger amplitude ($d_{11} < -0.2$) causes a strong variation of the corresponding real part due to the amplitude dependence of the soliton self-frequency shift.

These results clearly illustrate the qualitatively different effect of Hamiltonian (TOD) and dissipative (IRS) perturbations on the soliton bound states.

The numeric evaluation of the spectral data of L for each step of propagation of the soliton train also allows one to control the precision of the numerical procedure used to solve the NLSE [25].

Another possible effect of the strong perturbations is that the pulses taken initially to be one-soliton solutions of the NLS, may deform into the exact travelling-wave solutions of the perturbed NLS equation. Such effect has been reported in [26] due to the nonlinear gain and bandwidth limited amplification. Then the perturbed NLS equation goes int the Ginzburg-Landau (GL) equation whose stationary solutions possess characteristic phase modulation (chirp). It is due to this modulation that the soliton interaction reduces substantially.

6   Conclusions

Starting from the GKS model proposed in [6, 7, 9, 10] we have derived the perturbed CTC system describing the N-soliton train interaction of the perturbed NLS equation in the adiabatic approximation. For small perturbations the PCTC system is again completely integrable and provides us with an effective tool for analytic study of the asymptotic regimes of the N-soliton trains. In the non-adiabatic regime we propose a combined numeric solution of the NLS equation and the Zakharov-Shabat problem. Finally we mention that these methods can be applied also to the adiabatic interaction of the multicomponent NLS equation and its perturbed versions. Such equations describe the birefringence effects and soliton interactions in multi-mode fibers. We expect that their soliton interactions will be described by a generalized CTC-model in which the soliton phases δ_k are replaced by 'polarization' vectors $\vec{n}_k$, see [27]. These results can be used in soliton-based fiber-optics communications.

It is not difficult to treat also the perturbed sine-Gordon equation and derive the corresponding perturbed versions of (33). This and the study of the interactions of (anti-) solitons with breathers will be published elsewhere.

Finally we stress on the universal character of the CTC in the sense that it is independent on the dispersion law of the equation whose soliton interactions it describes.

Acknowledgements

A   The coefficients $Z_{\alpha,\beta}$.

Here we list the coefficients $Z_{\alpha,\beta}$ where Z takes the values N_k, M_k, Ξ_k and X_k while the pair of indices (α, β) is one of the following $(0,0)$, $(1,0)$ and $(0,1)$, see formula (21). With $c_{01} = c_{21} = d_{01} = 0$ (see the remark after eq. (20)) we have:

$$N_{00} = 4\nu_0 \left(\frac{c_{00}}{2} - c_{11}\mu_0 - \frac{2}{3}c_{20}(\nu_0^2 + 3\mu_0^2) + 8c_{31}\mu_0(\nu_0^2 + \mu_0^2) \right.$$
$$\left. + \frac{4}{3}\nu_0^2(d_{00} - d_{21}\mu_0) \right),$$

$$N_{10} = 2 \left(c_{00} - 2c_{11}\mu_0 + (8c_{31}\nu_0\mu_0 - 4c_{20})(\nu_0^2 + \mu_0^2) + 8\nu_0^2(d_{00} - d_{21}\mu_0) \right),$$

$$N_{01} = -4\nu_0 \left(c_{11} + 4c_{20}\mu_0 - 4c_{31}(\nu_0^2 + 3\mu_0^2) - \frac{4}{3}d_{21}\nu_0^2 \right),$$

$$M_{00} = -\frac{4}{3}\nu_0^2 \left(c_{11} + 4c_{20}\mu_0 - 12c_{31} \left(\mu_0^2 + \frac{7}{15}\nu_0^2 \right) + \frac{4}{5}\nu_0^2 d_{11} \right),$$

$$M_{10} = -\frac{8}{3}\nu_0 \left(c_{11} + 4c_{20}\mu_0 - 12c_{31}\mu_0^2 - \frac{56}{5}c_{31}\nu_0^2 + \frac{8}{5}d_{11}\nu_0^2 \right),$$

$$M_{01} = -\frac{4}{3}\nu_0^2 \left(c_{20} - 12c_{31}\mu_0 \right), \qquad \Xi_{00} = -c_{10} + 4c_{30}(\nu_0^2 + 3\mu_0^2) - \frac{2}{3}\nu_0^2 d_{10},$$

$$\Xi_{10} = \frac{4}{3}\nu_0 \left(6c_{30} - d_{10} \right), \qquad \Xi_{01} = 24c_{30}\mu_0,$$

$$X_{00} = c_{01} - 16c_{30}\mu_0(\nu_0^2 - \mu_0^2) + 4\nu_0^2 \left(\mu_0 d_{20} - \frac{1}{3}\mu_0 d_{10} \right),$$

$$X_{10} = -8\nu_0 \left(4c_{30}\mu_0 - d_{20}\mu_0 + \frac{1}{3}d_{10}\mu_0 \right),$$

$$X_{01} = 48\mu_0^2 c_{30} + 4\nu_0^2 \left(d_{20} - \frac{1}{3}d_{10} \right).$$

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Figure 1: Spatial evolution of (a) real and (b) imaginary parts of the eigenvalues of $2\mathrm{sech}\,(x)$ for different strengths of TOD: $c_{30} = 0.02$ (solid), $c_{30} = 0.022$ (dashed), $c_{30} = 0.03$ (dotted), $c_{30} = 0.05$ (dash-dotted).
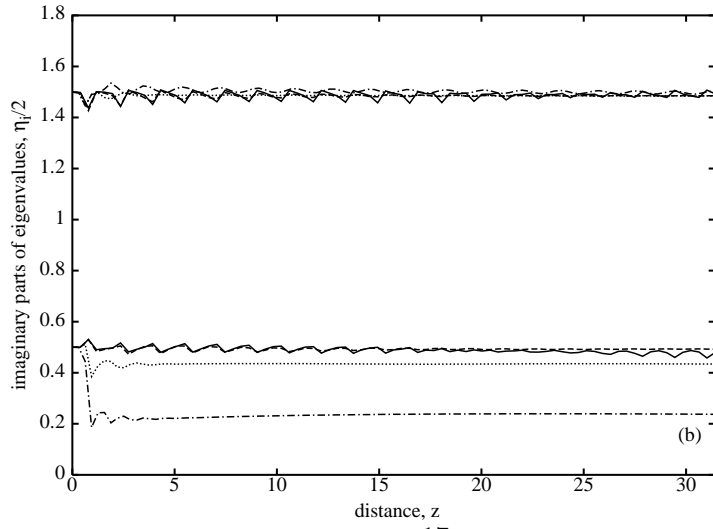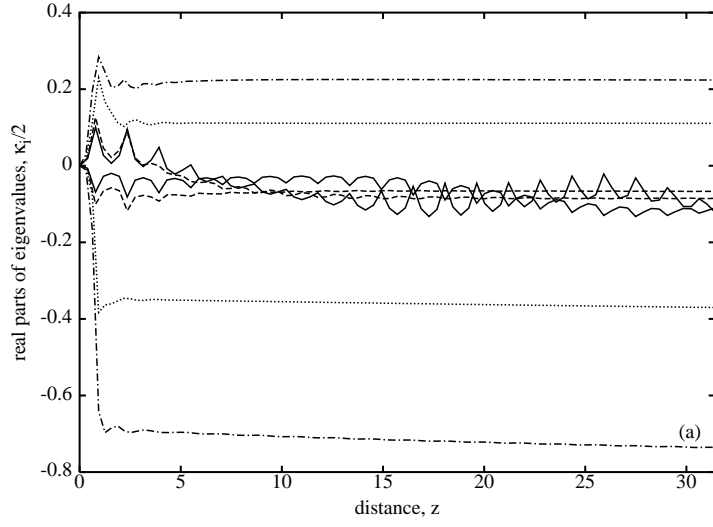
Figure 2: Spatial evolution of (a) real and (b) imaginary parts of the eigenvalues of $2\mathrm{sech}\,(x)$ for different strengths of IRS: $d_{11} = -0.0004$ (smal dashed), $d_{11} = -0.004$ (solid), $d_{11} = -0.02$ (small dash-dotted), $d_{11} = -0.04$ (dashed), $d_{11} = -0.4$ (dashed dotted).